COMP527
Data Mining and Visualisation
Problem Set 0

Danushka Bollegala

Question 1 Consider two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$ defined as $\boldsymbol{x} = (1, 2, -1)^\top$ and $\boldsymbol{y} = (-1, 0, 1)^\top$. Answer the following questions about these two vectors.

A. Compute the length (ℓ_2 norm) of $\boldsymbol{x}$ and $\boldsymbol{y}$. **(4 marks)**

$||\boldsymbol{x}||_2 = \sqrt{1+4+1} = \sqrt{6}$ *and* $||\boldsymbol{y}||_2 = \sqrt{1+0+1} = \sqrt{2}$

B. Compute the inner product between $\boldsymbol{x}$ and $\boldsymbol{y}$. **(2 marks)**

$\boldsymbol{x}^\top \boldsymbol{y} = -1 + 0 + -1 = -2$

C. Compute the cosine of the angle between the two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. **(4 marks)**

The definition of cosine similarity is $\frac{\boldsymbol{x}^\top \boldsymbol{y}}{||\boldsymbol{x}||_2 ||\boldsymbol{y}||_2}$. *Therefore, the required value will be* $-2/\sqrt{12}$.

D. Compute the Euclidean distance between the end points corresponding to the two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. **(4 marks)**

The definition of the Euclidean distance is $\sqrt{\sum_i (x_i - y_i)^2}$. *Therefore, we get* $\sqrt{4+4+4} = 2\sqrt{3}$

E. For any two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$ such that $||x||_2 = ||y||_2 = 1$ show that the following relationship holds between their cosine similarity $\cos(\boldsymbol{x}, \boldsymbol{y})$ and their Euclidean distance $\mathrm{Euc}(\boldsymbol{x}, \boldsymbol{y})$. **(6 marks)**

$$\mathrm{Euc}(\boldsymbol{x}, \boldsymbol{y})^2 = 2(1 - \cos(\boldsymbol{x}, \boldsymbol{y}))$$

$$\begin{aligned}
\mathrm{Euc}(\boldsymbol{x}, \boldsymbol{y})^2 &= (\boldsymbol{x} - \boldsymbol{y})^\top (\boldsymbol{x} - \boldsymbol{y}) \\
&= \boldsymbol{x}^\top \boldsymbol{x} + \boldsymbol{y}^\top \boldsymbol{y} - 2\boldsymbol{x}\boldsymbol{y} \\
&= 1 + 1 - 2\cos(\boldsymbol{x}, \boldsymbol{y}) \\
&= 2(1 - \cos(\boldsymbol{x}, \boldsymbol{y}) \quad \square
\end{aligned}$$

Question 2 Consider a matrix $\mathbf{A} \in \mathbb{R}^{2\times 2}$ defined as follows:

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Answer the following questions related to $\mathbf{A}$.

A. Compute the transpose $\mathbf{A}^\top$. **(2 marks)**

For a matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\mathbf{A}^\top = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. *Therefore, we have*

$$\mathbf{A}^\top = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

B. Compute the determinant $\det(\mathbf{A})$. **(2 marks)**

$\det(\mathbf{A}) = ac - bd = 2 \times 2 - 1 \times 1 = 3$

C. Compute the inverse $\mathbf{A}^{-1}$. **(4 marks)**

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

From which is follows,

$$\mathbf{A}^{-1} = \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix}.$$

D. Compute the eigenvalues and eigenvectors of $\mathbf{A}$. **(6 marks)**

Eigenvector $\boldsymbol{x}$ *corresponding to the eigenvalue* λ *satisfies the equation* $\mathbf{A}\boldsymbol{x} = \lambda\boldsymbol{x}$. *From which it follows that* $(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{x} = \mathbf{0}$. *Therefore,* $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. *In this case, we get* $\det\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = 0$. *Solving this second-order polynomial equation we get* $\lambda = 1, 3$, *which are the eigenvalues. Substituting these values separately in the eigenvalue equation we get the eigevectors corresponding* $\lambda = 1$ *and* $\lambda = 3$ *to be respectively* $(1, -1)^\top$ *and* $(1, 1)^\top$, *subjected to a scaling factor.*

Question 3

A. Given $\sigma(x) = \frac{1}{1+\exp(ax+b)}$, compute $\sigma'(x)$, the differential of $\sigma(x)$ with respect to x.

$\sigma'(x) = \frac{-a\exp(ax+b)}{(1+\exp(ax+b))^2}$

B. Given $H(p) = -p\log(p) - (1-p)\log(1-p)$, find the value of p that maximises $H(p)$.

$H'(p) = -\log(p) + \log(1-p) = 0$ *gives* $p = 0.5$

C. Find the maximum value of $g(x, y) = x^2 + y^2$ such that $y \leq -x + 1$.

Use Lagrange method of multipliers.

$$
\begin{aligned}
L(x, y, \lambda) &= x^2 + y^2 + \lambda(y + x - 1) \\
\frac{\partial L}{\partial x} &= 2x + \lambda = 0 \\
\frac{\partial L}{\partial y} &= 2y + \lambda = 0
\end{aligned}
$$

Substituting for x *and* y *we get*

$$
\begin{aligned}
L(\lambda) &= -\frac{\lambda^2}{2} - \lambda \\
\frac{\partial L}{\partial \lambda} &= -\lambda - 1 = 0 \\
\lambda &= -1
\end{aligned}
$$

Therefore, $x = y = 0.5$ *is the maximiser. Substituting these* $g(0.5, 0.5) = 0.5$*. Geometric solutions that measure the radius of the circle touching the line* $y = -x + 1$ *are also possible.*